M464 - Introduction To Probability II - Homework 7

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Chapter 5

Exercises:

1.3 Let X and Y be independent Poisson distributed random variables with parameters α and β, respectively. Determine the conditional distribution of X, given that $N = X + Y = n$.

Solution: We wish to compute $Pr\{X = k|X + Y = n\}$, for an arbitrary value of $k \in \mathbb{N}$. Note that by theorem 1.1 $X + Y \sim Pois(\alpha + \beta)$. We proceed as follow:

$$\begin{aligned}
Pr\{X = k|X + Y = n\} &= \frac{Pr\{X = k, X + Y = n\}}{Pr\{X + Y = n\}} && \text{def. of conditional prob.} \\
&= \frac{Pr\{X = k, Y = n - k\}}{Pr\{X + Y = n\}} && \text{replacing for the value of } X \\
&= \frac{Pr\{X = k\}Pr\{Y = n - k\}}{Pr\{X + Y = n\}} && \text{by independence of } X \text{ and } Y \\
&= \frac{\dfrac{e^{-\alpha}\alpha^k}{k!} \cdot \dfrac{e^{-\beta}\beta^{n-k}}{(n-k)!}}{\dfrac{e^{-(\alpha+\beta)}(\alpha+\beta)^n}{n!}} && \text{by def of Pois. distribution} \\
&= \frac{n! e^{-\alpha}\alpha^k e^{-\beta}\beta^{n-k}}{k!(n-k)! e^{-(\alpha+\beta)}(\alpha+\beta)^n} && \text{multiplying fractions} \\
&= \binom{n}{k}\frac{\alpha^k\beta^{n-k}}{(\alpha+\beta)^n} && \text{definition of binomial coefficient and cancelling } e'\text{s} \\
&= \binom{n}{k}\left(\frac{\alpha}{\alpha+\beta}\right)^k\left(\frac{\beta}{\alpha+\beta}\right)^{n-k} && \text{rearranging terms} \\
&= \binom{n}{k}p^k(1-p)^{n-k} && \text{letting } p = \frac{\alpha}{\alpha+\beta} \Longrightarrow 1 - p = \frac{\beta}{\alpha+\beta}
\end{aligned}$$

We recognize this distribution as a Binomial distribution with success probability p. Therefore,

$$X|X + Y = n \sim Binom\left(n, \frac{\alpha}{\alpha+\beta}\right)$$

Note that p is well defined because both α and β are greater than zero.

1.6 Messages arrive at a telegraph office as a Poisson process with mean rate of 3 messages per hour.

(a) What is the probability that no messages arrive during the morning hours 8:00 A.M to noon?

Solution: Let X = number of messages that arrive during the morning hours 8:00 A.M to noon. Then, by the properties of Poisson processes we know that

$$X \sim Pois(\frac{3}{hour} \cdot (12 - 8)hour) = Pois(12)$$

So now we can find $Pr\{X = 0\} = \dfrac{e^{-12}12^0}{0!} = e^{-12} = 0.00000614421$. This is very unlikely, which makes sense because on average 3 messages arrive per hour and we are looking at a period of 4 hours with no messages arriving.

(b) What is the distribution of the time at which the first afternoon message arrives?

Solution: Let X be the poisson process and let T = the time at which the first afternoon message arrives. Afternoon is the period between 12:00 p.m. and 12:00 a.m. We know the distribution of messages arriving in this period and so we can compute the distribution of time, for $t = 13, 14, 15, \ldots 24$ as follow:

$$\begin{aligned} Pr\{T > t\} &= Pr\{\text{the first afternoon message arrives after } t \text{ units of time}\} \\ &= Pr\{(X(t) - X(12)) = 0\} \\ &= Pr\{(X(t-12)) = 0\} \qquad \text{By properties of Pois. process} \end{aligned}$$

We know the distribution of $X(t-12) \sim Pois(\frac{3}{hour} \cdot (t-12)hours) = Pois(3(t-12))$. Hence,

$$Pr\{T > t\} = Pr\{(X(t-12)) = 0\} = \frac{e^{-3(t-12)}(3(t-12))^0}{0!} = e^{-3(t-12)}$$

Finally, to get the cumulative distribution take the complement of the survival function:

$$Pr\{T \leq t\} = 1 - Pr\{T > t\} = 1 - e^{-3(t-12)}$$

Since $t > 12$, a change of variables $t - 12 = x \Longrightarrow Pr\{T \leq x\} = 1 - e^{-3x}$, thus, $T \sim Exp(3)$.

Problems:

1.2 Suppose that minor defects are distributed over the length of a cable as a Poisson process with rate α, and that, independently major defects are distributed over the cable according to a Poisson process of rate β. Let $X(t)$ be the number of defects, either major or minor, in the cable up to length t. Argue that $X(t)$ must be a Poisson process of rate $\alpha + \beta$.

Solution:

Let us check that $\langle X(t); t \geq 0\rangle$ is a Poisson process of intensity (or rate) $\alpha + \beta$. First, let us define $\langle Y(t); t \geq 0\rangle$ to be the Poisson process for minor defects and $\langle Z(t); t \geq 0\rangle$ to be the Poisson process for major defects. Then, by definition of Poisson process we know that $Y(t) \sim Pois(\alpha t)$, and $Z(t) \sim Pois(\beta t)$, both for every $t > 0$. Now, by definition, the total number of defects is the sum of minor and major defects, i.e., $X(t) = Y(t) + Z(t)$. Since Y and Z are independent, by theorem 1.1 we conclude $X(t) = Y(t) + Z(t) \sim Pois((\alpha + \beta)t)$, which holds for every $t > 0$. Also, $\alpha > 0$ and $\beta > 0$ (by definition of Poisson process), and so $\alpha + \beta > 0$. This takes care of conditions (i) and (v) given in class for being a Poisson process. For condition (ii) note that $Y(t) \in \mathbb{N}$ and $Z(t) \in \mathbb{N}$ and thus, $X(t) \in \mathbb{N}$. Condition (iv) is easily checked: $X(0) = Y(0) + Z(0) = 0 + 0 = 0$. It remains to check condition (iii) of independent stationary increments. Let us check this in two steps:

a) Independent increments: Choose arbitrary time points t_i. Then,

$$X(t_{k+1}) - X(t_k) = [Y(t_{k+1}) + Z(t_{k+1})] - [Y(t_k) + Z(t_k)] = [Y(t_{k+1}) - Y(t_k)] + [Z(t_{k+1}) - Z(t_k)]$$

Since Y and Z are Poisson processes, each summand is independent by the independent of increments of each process. Also, since Y and Z are independent, their sum is independent, which shows that X has independent increments.

b) Stationary increments: let us show that for any $t > 0$, the distribution of $X(s+t) - X(s)$ does not depend on s.

$$\begin{aligned} Pr\{X(s+t) - X(s) = k\} &= Pr\{[Y(s+t) + Z(s+t)] - [Y(s) + Z(s)] = k\} \text{ by definition of } X \\ &= Pr\{[Y(s+t) - Y(s)] + [Z(s+t) - Z(s)] = k\} \text{ rearranging terms} \\ &= \sum_{n=0}^{k} Pr\{[Y(s+t) - Y(s)] + [Z(s+t) - Z(s)] = k | [Y(s+t) - Y(s)] = n\} Pr\{[Y(s+t) - Y(s)] = n\} \\ &\quad \text{(law of total prob)} \\ &= \sum_{n=0}^{k} Pr\{[Z(s+t) - Z(s)] = k - n\} Pr\{[Y(s+t) - Y(s)] = n\} \text{ by independence of } Y \text{ and } Z \end{aligned}$$

Since both Y and Z have stationary, independent increments, the distribution of each product above does not depend on s and so the distribution of X won't depend on s either, i.e, X has independent stationary increments.

1.3 The *generating function* of a probability mass function $p_k = Pr\{X = k\}$, for $k = 0, 1, \ldots$, is defined by

$$g_X(s) = E[s^X] = \sum_{k=0}^{\infty} p_k s^k \qquad \text{for } |s| < 1$$

Show that the generating function for a Poisson random variable X with mean μ is given by

$$g_X(s) = e^{-\mu(1-s)}$$

Solution: Let $X \sim Pois(\mu)$ and $|s| < 1$. Then,

$$\begin{aligned}
g_X(s) &= \sum_{k=0}^{\infty} p_k s^k && \text{by definition of generating function} \\
&= \sum_{k=0}^{\infty} \frac{e^{-\mu}\mu^k}{k!} s^k && \text{Poisson p.m.f} \\
&= e^{-\mu} \sum_{k=0}^{\infty} \frac{(s\mu)^k}{k!} && \text{Factoring constant and rearranging terms} \\
&= e^{-\mu} e^{(s-\mu)} && \text{Taylor series of } e \\
&= e^{-\mu(1-s)} && \text{Summing exponents}
\end{aligned}$$

1.6 Let $\{X(t); t \geq 0\}$ be a Poisson process of rate λ. For $s, t > 0$, determine the conditional distribution of $X(t)$, given that $X(t+s) = n$.

Solution: Let $k \leq m$. Then:

$$\begin{aligned}
Pr\{X(t) = k | X(t+s) = n\} &= \frac{Pr\{X(t) = k, X(t+s) = n\}}{Pr\{X(t+s) = n\}} && \text{conditional prob.} \\
&= \frac{Pr\{X(t+s) = n | X(t) = k\} Pr\{X(t) = k\}}{Pr\{X(t+s) = n\}} && \text{conditional prob.} \\
&= \frac{Pr\{X(t+s) - X(t) = n-k\} Pr\{X(t) = k\}}{Pr\{X(t+s) = n\}} && \text{Independent increments of Pois. process}
\end{aligned}$$

Now, we know the distribution of each of these:

$$X(t+s) - X(t) \sim Pois([(t+s) - t]\lambda) = Pois(\lambda s); \quad X(t+s) \sim Pois(\lambda(t+s)); \quad X(t) \sim Pois(\lambda t)$$

Hence, we can compute the distribution we are interested in:

$$\begin{aligned}
Pr\{X(t) = k | X(t+s) = n\} &= \frac{\frac{e^{-\lambda s}(\lambda s)^{n-k}}{(n-k)!} \cdot \frac{e^{-\lambda t}(\lambda t)^k}{k!}}{\frac{e^{-\lambda(t+s)}[\lambda(t+s)]^n}{n!}} = \frac{e^{-\lambda s}(\lambda s)^{n-k} e^{-\lambda t}(\lambda t)^k n!}{e^{-\lambda(t+s)} k!(n-k)![\lambda(t+s)]^n} \\
&= \binom{n}{k} \frac{(\lambda s)^{n-k}(\lambda t)^k}{\lambda^n (t+s)^n} \\
&= \binom{n}{k} \frac{s^{n-k} t^k}{(t+s)^n} \\
&= \binom{n}{k} \left(\frac{t}{t+s}\right)^k \left(\frac{s}{t+s}\right)^{n-k} \\
&= \binom{n}{k} p^k (1-p)^{n-k} && \text{Letting } p = \frac{t}{t+s} \Rightarrow 1 - p = \frac{s}{t+s}
\end{aligned}$$

Hence, $X(t)|X(t+s) = n \sim Binom\left(n, \frac{t}{t+s}\right)$